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THE WAVE DRAG OF HIGHLY SWEPT WINGS; A COMPARISON OF LINEAR THEORY AND SLENDER BODY THEORY

by

T. NONWEILER, B.Sc.

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The Wave Drag of Highly-Swept Wings - A Comparison of Linear Theory and Slender Body Theory.

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SUMMARY

This note comments on the comparison between the answer obtained by linear theory for the wave drag of slender-wings (as interpreted by the limit $\sqrt{M^2-1}\cot\Lambda\to 0$) and that value for the drag obtained by Slender-Body Theory. It is shown that for fully tapered wings the agreement is exact, and that there is reason to suppose that the same is true for all wing planforms, unless the trailing-edge is unswept, or the wing section has a finite trailing-edge thickness. Some remarks are included concerning the drag of slender delta wings.

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1. This note is in the nature of an addendum to an earlier one[1] discussing the formulation of an expression, on the basis of the linearised theory of compressible flow, for the wave drag at zero incidence of a sweptback fully tapered wing of arbitrary section. As a particular example of the method developed therein, the drag of such wings of slender planform was considered as a particular case, and a few numerical results evaluated.

It has subsequently been pointed out to the author that these calculations are in close agreement with numerical estimates of the drag of such wings calculated on the basis of the " Slender Body" Theory advanced by Ward some years ago.[2] The present note discusses the comparison of the two theories on a formal basis.

2. This agreement noticed by other investigators is, in fact, an exact one; this is shown in the Appendix of this note by a straightforward mathematical argument. The so-called " Slender Body" Theory yields the answer[2] that the drag D is given by

$$\frac{D}{q} = \frac{1}{2\pi L^2} \int_0^1 \int_0^1 S''(\xi) S''(\eta) \, \ell n \left| \frac{1}{\xi - \eta} \right| d\xi \, d\eta$$

subject to certain reservations (which we shall detail subsequently), where $S(\xi)$ is the cross-sectional area of the body (or wing) at a distance ξL downstream of its nose (or apex), - L being the overall length of the body. On the other hand the linear theory yields a relation of the kind :

$$\frac{D}{q} = s^2 \tau^2 f \left(A \tan \Lambda_o, \sqrt{M^2-1} \cot \Lambda_o, \Sigma\right)$$

where s is the wing semi-span, τ the thickness/chord ratio, A the wing aspect ratio and Λ_o the angle of sweepback of the leading-edge. The parameter Σ is used here to describe the section shape. In reference 1, the drag of "slender" wings was stated as

$$\frac{D}{q} = s^2 \tau^2 \lim_{\sqrt{M^2-1} \cot \Lambda_o \to 0} \left\{ f \left(A \tan \Lambda_o, \sqrt{M^2-1} \cot \Lambda_o, \Sigma\right) \right\}$$

$$= s^2 \tau^2 f(A \tan \Lambda_o, 0, \Sigma).$$

The slender body theory gives, of course, a very similar form of answer :

$$\frac{D}{q} = s^2 \tau^2 \phi \left(\frac{L}{c}, \Sigma\right), \text{ say,}$$

where c is the wing chord; and the two functions f and ϕ are shown, in the Appendix, to be identical.

3. This deduction alludes, of course, only to the fully-tapered wings. It is strong support for the argument, however, that the "Slender Body" Theory gives the appropriate limit of the linear theory for the general case of the highly-swept wing, whatever its planform. The correspondence is by no means obvious from first principles, as the "Slender Body" Theory uses the momentum theorem, in conjunction with a quadratic approximation to the pressure, to evaluate

the drag force, - the potential of the flow, however, being obtained from the linearised equation of motion. In this sense, the "Slender Body" Theory may be said to be more exact than the linearised theory. However, a scrutiny of the analysis of reference 2 shows that the quadratic terms in the pressure equation do not contribute to the expression for the drag given above, but reveal themselves in additional terms which are in general zero for wings with closed sections. Thus, the agreement with linear theory is not surprising : without the quadratic terms, the momentum integral for the drag, as well as the flow potential, have both a "linear" form.

4. To enlarge on this effect of the second-order terms, it may be mentioned that the pressure equation is used in "Slender Body" Theory to find the momentum flux through a finite plane area, perpendicular to the flow direction, and embracing the body at the furthest downstream cross-section of the body (or wing). The pressure there depends on the local value of the cross-flow potential ϕ_o, and its change over this plane (i.e. the derivatives of ϕ_o with respect to r and θ, say, the polar co-ordinates in this plane $\xi=1$); and in "Slender-Body" theory, the appropriate second-order terms in velocities included in the expression for pressure involve $\left(\frac{\partial\phi_o}{\partial r}\right)^2$ and $\frac{1}{r^2}\left(\frac{\partial\phi_o}{\partial\theta}\right)^2$. Now, provided the trailing-edge is swept and the wing section is a closed one, the value of ϕ_o over this plane ($\xi=1$) is a constant (given by b_o in the analysis of reference 2); this may easily be shown, as the disturbance to the cross-flow vanishes at this plane. The boundary conditions at the wing surface normally require a line distribution of

sources and sinks to obtain the appropriate cross-flow in the transverse planes ξ = constant, and this distribution tends to vanish as $\xi \rightarrow 1$. Since ϕ_o is a constant, it will be evident that both $\frac{\partial \phi_o}{\partial r}$ and $\frac{\partial \phi_o}{\partial \theta}$ must be zero, in the plane $\xi = 1$, and so there is no contribution to the pressure at that plane from the quadratic velocity terms. That is why there is, for the general case of the wing, nothing incompatible between "Slender Body" theory, and one in which the linearised form is used throughout. Consequently, agreement between the two theories is to be expected where they meet - that is, for slender wings.

5. The exceptions noted above bear on important cases. The wave drag of wings with a finite trailing-edge thickness has not received a great deal of attention, but the linearised theory may not give the same answer for the slender wing as the theory of Ward ; in this case, the latter theory could be said to be more exact. The other exception, which includes all wings having a straight trailing-edge but a highly swept leading-edge - that is, all wings which have a slender "cropped" (or "uncropped") delta planform - is of great interest, but as we shall see is a case which does raise some difficulty in interpretation. In both exceptions there are terms in the expression for the drag given by "slender-body" theory which are additional to that given in para. 2 and which are quoted at the beginning of the Appendix.

6. For the delta wing (with a finite trailing-edge angle) it will be found that the variation of $S''(\xi)$, in the double integral for the drag quoted earlier, has a logarithmic

singularity at $\xi = 1$, but nevertheless the integral remains convergent, and so yields a finite contribution to the drag. Of the remaining two terms (as given in equation (1) of the Appendix) the term involving $S'(1)$ remains finite, as $S'(\xi)$ has a finite limit at $\xi = 1$ (the trailing-edge), but the last term, though again finite, involves the expression $\ell n(1/\sqrt{M^2-1}\cot\Lambda_o)$ and the complete expression for the drag is of the form:

$$\frac{D}{q} = s^2 \tau^2 \phi\,(\Sigma) + \text{constant} \times \left[\frac{S'(1)}{c}\right]^2 \ell n\left(\frac{1}{\sqrt{M^2-1}\cot\Lambda_o}\right).$$

The question arises as to the compatibility of this result with the answer obtained by the same theory for "near-delta" wings - i.e. those with values of $\sigma = \frac{1}{4} A \tan \Lambda_o = \frac{c}{L}$ which are a little greater than unity.

7. For such wings, "Slender Body" theory has been shown to be in agreement with linearised theory, from which it is found in reference 1 that, as $\sigma = \frac{c}{L} \rightarrow 1$ from above,

$$\frac{D}{q} \sim s^2 \tau^2 \ell n\left(\frac{1}{\sigma-1}\right)$$

i.e. the drag becomes infinite. How is this result reconcilable with the finite drag of the slender delta wing?

This question is easily resolved if the well-known result of linearised theory is recalled, that for a delta wing (or its reverse)

$$\frac{D}{q} \sim s^2 \ \tau^2 \ \ell n \frac{1}{\sqrt{M^2-1} \ \cot \Lambda_o} \quad \text{as } \sqrt{M^2-1} \cot \Lambda_o \rightarrow 1$$

- this result being well-known in the sense that it will be readily agreed that the drag of delta wing is (by this theory) infinite "at M = 1". It also implies that (D/qs^2) is infinitely large for a slender wing. Now the results of linearised theory for the slender and the "near-delta" wing, are both of course compatible with one another, and only differ because of the reversal of the order of taking the limits in the general expression for the drag, - allowing first, either the wing to become slender, or the trailing-edge sweep to vanish. Viewed in the same light, there is therefore nothing incompatible about the answer given by "Slender-Body" theory for a delta wing, despite the infinite drag of a wing with $\frac{L}{c} = 1+$, since we see that it is at least qualitatively similar to that of linearised theory. The value of (D/qs^2) is certainly infinitely large for a slender delta wing, and the finite expression derived for it merely shows the nature of this singularity.

8. Conclusions

(i) That the drag of fully-tapered swept wings calculated by linear theory in the limiting condition $\sqrt{M^2-1} \cot \Lambda_o \rightarrow 0$ is identical with the result of "Slender Body" Theory for such wings.

(ii) That any difference between the drag calculated by "Slender Body" theory and linear theory could only arise because of the use of a quadratic approximation to the pressure employed in the former.

(iii) That the quadratic terms in the pressure equation do not contribute to the drag of swept wings, in general, so that the agreement noted above for fully-tapered wings should extend to all planforms.

(iv) That the exceptions to this rule are those wings which have a section with a finite trailing-edge thickness, or a planform with an unswept trailing-edge (i.e. a delta wing). For these wings, the expression for the drag given by Slender Body theory involves a contribution from the quadratic terms, and so would be expected to differ from the answer found by linearised theory.

(v) That the expression for the drag of slender, cropped (or uncropped) delta wings, is (if not quantitatively) at least qualitatively identical in both theories.

References

1. T. Nonweiler — The Theoretical Wave Drag at Zero Lift of Fully Tapered Swept Wings of Arbitrary Planform.
   C. of A. Report No. 76 (1953)

2. G. N. Ward — Supersonic Flow past Slender Pointed Bodies.
   Quart. J. Mech. and App. Maths. Vol. II Pt. 1 (1949)

Appendix

The fundamental result of the Slender Body Theory of Ward[2] in relation to the wave drag at zero incidence (D, say) is that

$$L^2\left(\frac{D}{q}\right) = \frac{1}{2\pi}\int_0^1\int_0^1 S''(\xi)S''(\eta)\,\ln\left|\frac{1}{\xi-\eta}\right| d\xi d\eta - \frac{S'(1)}{2\pi}\int_0^1 S''(\xi)\,\ln\left|\frac{1}{1-\xi}\right| d\xi$$

$$-\left(\int_C \phi_0 \frac{\partial\phi_0}{\partial\nu}\, d\tau\right)_{\xi=1} \qquad \ldots (1)$$

where $S(\xi)$ is the cross-sectional area of the body at a distance ξL downstream of its nose or apex (L being the body overall length), ϕ_0 is the limiting form of the perturbation potential of the flow on or near the body, and ν and τ are orthogonal co-ordinates in cross-sectional planes, ν being measured normal to the body and τ around it; the contour C is described around the cross-sectional perimeter of the body. The potential ϕ_0 is shown in reference 2 to be derived from that for the cross-flow in planes ξ = constant, which satisfies the boundary condition

$$\frac{\partial\phi_0}{\partial\nu} = \frac{d\nu}{d\xi} \quad \text{on} \quad C \qquad \ldots (2)$$

i.e. on the body surface.

We suppose now that the body in question is a fully-tapered swept wing, with a swept-back trailing edge, and with a leading edge angle of sweepback given by $\cot^{-1}m_0$, say. The cross-sectional area of such a body is given by

$$S(\xi) = 4\int_0^{m_0\xi L} z_0\, dy \qquad \ldots (3)$$

where z_o is the wing semi-ordinate, and y is a co-ordinate measured spanwise from the wing root chord line, so that $y = \pm m_o \xi L$ describes the position of the wing leading edge. We further suppose that the wing section is similar over the entire span and is given by

$$z_o \Big|_{y=0} = \frac{t}{2} Z\left(\frac{\xi L}{C}\right) \qquad \dots (4)$$

along the root chord line. We define $Z(k)$ as zero for $k < 0$, and $k > 1$. Then it follows from simple geometry that, in general, if s is the wing semi-span (so that $s = m_o L$),

$$\left.\begin{aligned} z_o &= \frac{t}{2}\left(1 - \frac{y}{s}\right) Z(k) \\ \text{where } k &= \left(\xi L - \frac{y}{m_o}\right)\Big/\left(1 - \frac{y}{s}\right)c \end{aligned}\right\} \qquad \dots (5)$$

Substituting from (5) in (3), it follows that

$$S(\xi) = 2t\int_0^{m_o L\xi} \left(1 - \frac{y}{s}\right) Z\left[\frac{\xi L - \frac{y}{m_o}}{c\left(1 - \frac{y}{s}\right)}\right] dy$$

This integral can be simplified by changing the variable of integration from y to k, as defined in (5): for then

$$\left(1 - \frac{y}{s}\right) = \left(\frac{\frac{s}{m_o} - \xi L}{\frac{s}{m_o} - kc}\right) \quad \text{and} \quad \frac{dy}{dk} = sc\,\frac{\left(\xi L - \frac{s}{m_o}\right)}{\left(\frac{s}{m_o} - kc\right)^2}$$

Further, if we put

$$\frac{1}{4} A \tan \Lambda_o = \frac{s}{m_o c} = \frac{L}{c} = \sigma, \text{ say} \qquad \ldots (6)$$

noting that $\sigma > 1$ is the condition that the trailing-edge is sweptback (as already assumed), our expression for the cross-sectional area simplifies to

$$S(\xi) = 2st\sigma^2 \ (1-\xi)^2 \int_o^{\sigma\xi} \frac{Z(k)}{(\sigma-k)^3} \, dk \qquad \ldots (7)$$

Evidently $S(\xi)$ is continuous and vanishes at $\xi = 1$. Further we find by differentiation that

$$S'(\xi) = 2st \frac{Z(\sigma\xi)}{(1-\xi)} - 4st\sigma^2(1-\xi) \int_o^{\sigma\xi} \frac{Z(k)}{(\sigma-k)^3} \, dk$$

or after integrating by parts

$$S'(\xi) = 2st\sigma^2(1-\xi) \int_o^{\sigma\xi} \frac{Z'(k)}{(\sigma-k)^2} \, dk \qquad \ldots (8)$$

from which it appears that $S'(\xi)$ is also continuous, and vanishes at $\xi=0$, and at $\xi=1$ (provided that $\sigma > 1$). Again differentiating, if follows that

$$S''(\xi) = \frac{2st\sigma Z'(\sigma\xi)}{1-\xi} - 2st\sigma^2 \int_o^{\sigma\xi} \frac{Z'(k)}{(\sigma-k)^2} \, dk \qquad \ldots (9a)$$

If $Z'(k)$ is of such a form as to allow an integration by parts, this has the alternative expression

$$\left.\begin{aligned} S''(\xi) &= 2st\sigma\left[Z'(\sigma\xi) + \int_0^{\sigma\xi} \frac{kZ''(k)}{\sigma-k}\,dk\right] \text{ for } 0 < \xi < \frac{1}{\sigma} \\ &= 2st\sigma\left[-\frac{Z'(1)}{\sigma-1} + \int_0^{1} \frac{kZ''(k)}{\sigma-k}\,dk\right] \text{ for } \frac{1}{\sigma} < \xi < 1 \end{aligned}\right\} \quad \ldots (9b)$$

We see that $S''(\xi)$ will be generally discontinuous since $Z'(k)$ will be discontinuous in general for $0 < k < \sigma$. Even if $Z'(k)$ is continuous in the range $0 < k < 1$, if the wing section has a finite leading-edge angle, $Z'(k)$ and so $S''(\xi)$ has at least one discontinuity at $k = 1$ and $\xi = \frac{1}{\sigma}$.

Now referring back to equation (1) we see that the second term on the right-hand side vanishes, since by (8), $S'(1) = 0$. The third term involves a knowledge of ϕ_o at the furthest downstream cross-section $\xi=1$. In general at any plane ξ = constant, ϕ_o must be found so that, from (2),

$$\left.\frac{\partial\phi_o}{\partial z}\right|_{z=0} = \left.\frac{d\nu}{d\xi}\right|_{z=0} = (\operatorname{sgn} z)\frac{dz_o}{d\xi} = \frac{Lt}{2c}Z'\left[\frac{\xi L-\frac{y}{m_o}}{c(1-\frac{y}{s})}\right].(\operatorname{sgn} z)$$

where we have used equation (5) to interpret $\frac{dz_o}{d\xi}$. Such a boundary condition is met by a distribution of sources along $z = 0$. It is seen that on $\xi=1$,

$$\left.\frac{\partial\phi_o}{\partial z}\right|_{z=0} = \frac{Lt}{2c}Z'(\sigma).(\operatorname{sgn} z) = 0, \text{ except at } y = \pm m_o L = \pm s\ ;$$

and consequently the sources vanish at $\xi = 1$, and ϕ_o is as a result independent of y and z in this plane. Thus the third term on the right-hand side of (1) also vanishes. Evidently then

$$\frac{D}{q} = \frac{1}{2\pi} \int_0^1 \int_0^1 S''(\xi) S''(\eta) \; \ell n \left|\frac{1}{\xi-\eta}\right| \; d\xi d\eta \qquad \ldots (10)$$

where $S''(\xi)$ is given by equation (9) in terms of Z'.

Using the first form of equation (9) (a), it follows then in (10) that

$$\frac{D}{qt^2} = \frac{2\sigma^2 s^2}{\pi L^2} \int_0^1 \int_0^1 \left[\frac{Z'(\sigma\xi)}{1-\xi} - \sigma \int_0^{\sigma\xi} \frac{Z'(k)}{(\sigma-k)^2} dk\right] \left[\frac{Z'(\sigma\eta)}{1-\eta} - \sigma \int_0^{\sigma\eta} \frac{Z'(\ell)}{(\sigma-\ell)^2} d\ell\right] \ell n \left|\frac{1}{\xi-\eta}\right| d\xi d\eta$$

or changing the variables of integration from ξ and η to $k = \sigma\xi$, and $\ell = \sigma\eta$, we have on expansion that

$$\frac{D}{qm^2t^2} = \frac{2\sigma^2}{\pi} \int_0^\sigma \int_0^\sigma \left[\frac{Z'(k)Z'(\ell)}{(\sigma-k)(\sigma-\ell)} - \frac{Z'(\ell)}{(\sigma-\ell)} \int_0^k \frac{Z'(p)}{(\sigma-p)^2} dp - \frac{Z'(k)}{(\sigma-k)} \int_0^\ell \frac{Z'(q)}{(\sigma-q)^2} dq \right.$$

$$\left. + \int_0^k \frac{Z'(p)}{(\sigma-p)^2} dp \int_0^\ell \frac{Z'(q)}{(\sigma-q)^2} dq \right] \ell n \left|\frac{\sigma}{k-\ell}\right| dk d\ell. \qquad \ldots (11)$$

We now attempt to cast the integrand into the form

$$\frac{D}{qm^2t^2} = \frac{\sigma^2}{\pi} \int_0^1 \int_0^1 Z'(k) \; Z'(\ell) \; f(k,\ell) \; dk \; d\ell$$

but in doing so certain care has to be exercised as the integrand is discontinuous along the lines $k = 1$, and $\ell = 1$. Taking the component terms in the expansion of the integral in (11) term by term, we note that

$$\int_0^\sigma\int_0^\sigma \frac{Z'(k)Z'(\ell)}{(\sigma-k)(\sigma-\ell)} \ln\left|\frac{\sigma}{k-\ell}\right| dk\,d\ell = \int_0^1\int_0^1 \frac{Z'(k)Z'(\ell)}{(\sigma-k)(\sigma-\ell)} \ln\left|\frac{\sigma}{k-\ell}\right| dk\,d\ell \qquad \ldots(12)$$

since $Z'(k) = 0$ for $k > 1$. Again

$$\int_0^\sigma\int_0^\sigma \frac{Z'(\ell)}{(\sigma-\ell)}\left[\int_0^k \frac{Z'(p)}{(\sigma-p)^2}dp\right] \ln\left|\frac{\sigma}{k-\ell}\right| dk\,d\ell = \int_0^1 \frac{Z'(\ell)}{(\sigma-\ell)}d\ell\int_0^\sigma \ln\left|\frac{\sigma}{k-\ell}\right| dk\int_0^k \frac{Z'(p)}{(\sigma-p)^2}dp$$

$$= \int_0^1 \frac{Z'(\ell)}{\sigma-\ell}\,d\ell\int_0^\sigma \frac{Z'(p)}{(\sigma-p)^2}\,dp\int_p^\sigma \ln\left|\frac{\sigma}{k-\ell}\right| dk$$

$$= \int_0^1 d\ell\int_0^1 \frac{Z'(\ell)Z'(k)}{(\sigma-\ell)(\sigma-k)^2}\left[\int_k^\sigma \ln\left|\frac{\sigma}{p-\ell}\right| dp\right] dk\,d\ell$$

where we have first changed the order of integration between p and k, and then interchanged these symbols. Evaluating the integral with respect to p, we find that

$$\int_0^\sigma\int_0^\sigma \frac{Z'(\ell)}{(\sigma-\ell)}\left[\int_0^k \frac{Z'(k)}{(\sigma-p)^2}dp\right] \ln\left|\frac{\sigma}{k-\ell}\right| dk\,d\ell = \int_0^1\int_0^1 \frac{Z'(k)\ Z'(\ell)}{(\sigma-\ell)(\sigma-k)}\left[\frac{\sigma-\ell}{\sigma-k}\ln\left|\frac{\sigma}{\sigma-\ell}\right| + \frac{\ell-k}{\sigma-k}\ln\left|\frac{\sigma}{k-\ell}\right| + 1\right] dk\,d\ell \qquad \ldots(13)$$

and similarly

$$\int_0^\sigma\int_0^\sigma \frac{Z'(k)}{\sigma-k}\left[\int_0^\ell \frac{Z'(q)}{(\sigma-q)^2}\,dq\right]\ell n\left|\frac{\sigma}{k-\ell}\right|\,dk\,d\ell$$

$$=\int_0^1\int_0^1 \frac{Z'(k)Z'(\ell)}{(\sigma-\ell)(\sigma-k)}\left[\frac{\sigma-k}{\sigma-\ell}\ell n\left|\frac{\sigma}{\sigma-k}\right| + \frac{k-\ell}{\sigma-\ell}\ell n\left|\frac{\sigma}{k-\ell}\right| + 1\right]dk\,d\ell\ . \qquad \ldots\ (14)$$

It will be seen that equation (14) enables the repeated integrals of the second and third terms inside the square brackets in equation (11) to be represented in the form required. To put the last term in the brackets in a similar form, we proceed with a similar series of manipulations to those used in deriving equation (14), and consequently find that

$$\int_0^\sigma\int_0^\sigma\left[\int_0^k \frac{Z'(p)}{(\sigma-p)^2}\,dp\int_0^\ell \frac{Z'(q)}{(\sigma-q)^2}\,dq\right]\ell n\left|\frac{\sigma}{k-\ell}\right|\,dk\,d\ell$$

$$=\int_0^1 d\ell\int_0^1 \frac{Z'(k)Z'(\ell)}{(\sigma-\ell)^2(\sigma-k)^2}\left[\int_k^\sigma dp\int_\ell^\sigma \ell n\left|\frac{\sigma}{p-q}\right|dq\right]dk$$

$$=\int_0^1\int_0^1 \frac{Z'(k)Z'(\ell)}{(\sigma-\ell)(\sigma-k)}\left[\frac{\sigma-\ell}{2(\sigma-k)}\,\ell n\left|\frac{\sigma}{\sigma-\ell}\right| + \frac{\sigma-k}{2(\sigma-\ell)}\ell n\left|\frac{\sigma}{\sigma-k}\right| - \frac{(k-\ell)^2}{2(\sigma-\ell)(\sigma-k)}\,\ell n\left|\frac{\sigma}{k-\ell}\right| + \frac{3}{2}\right]dk\,d\ell \qquad \ldots\ (15)$$

Collecting the results of equations (12) to (15), and using them in (11), we have

$$\frac{D}{qm^2t^2} = \frac{\sigma^2}{\pi}\int_0^1\int_0^1 Z'(k)Z'(\ell)\left[\frac{1}{(\sigma-\ell)^2}\ell n\left|\frac{\sigma-k}{k-\ell}\right| + \frac{1}{(\sigma-k)^2}\ell n\left|\frac{\sigma-\ell}{\ell-k}\right| - \frac{1}{(\sigma-\ell)(\sigma-k)}\right]dk\,d\ell. \qquad \ldots\ (16)$$

This is the form of the expression deduced in reference 1, with $\sigma = \frac{1}{4}A\tan\Lambda_o$.